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ON STRONGLY S-PROJECTIVE MODULES

OUSSAMA AYMANE ES SAFI, NAJIB MAHDOU, AND ÜNSAL TEKİR

ABSTRACT. Let R be a commutative ring with identity and S a multiplicative subset of R. In this work, we define and study new versions of the u-S-projective module and the u-S-hereditary ring called strongly S-projective module and strongly S-hereditary ring respectively. In this work, a module M is called strongly S-projective if there exists a projective submodule P of M such that $sM \subseteq P$. Several properties concerning those notions are shown in this study. The exploration of the relationship between the introduced notions and the u-S-projectivity, projectivity, and other classical ones led to important results. For instance, we showed that any strongly S-projective module is u-S-projective.

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1. INTRODUCTION

This paper assumes all rings are commutative with identity and all modules are nonzero unital. For completeness, we begin with some definitions and notations used in this paper. For a ring R, we will denote by $U(R)$, $reg(R)$, $Z(R)$ and $idem(T)$ (T is a subset of R) the set of unit elements, regular elements, zero-divisor elements, and idempotent elements of T, respectively. Recently, motivated by the work of Anderson and Dumitrescu, S-versions of some classical notions have been introduced (see, for instance, [1], [3], [5], [9], [10], [12], [13], [16]). Some works dealt with the S-version of projective and injective notions. For instance, in [17] the authors define u-S-projective ideals using u-S-exact sequences. More precisely, in this work, an R-module P is called u-S-projective if for every u-S-short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced sequence $0 \to Hom(P, A) \xrightarrow{f} Hom(P, B) \xrightarrow{g} Hom(P, C) \to 0$ is u-S-exact. Inspired by the definition given the S-finite module, we define the notion of strongly S-projective as follows: an R-module M is called strongly S-projective if there exist an element s of S and a projective R-module between sM and M. We also define a strongly S-hereditary ring as a ring in which every ideal is strongly S-projective for a multiplicative set S. From these definitions, we conclude that a projective module is strongly S-projective and a hereditary ring is strongly S-hereditary for every multiplicative subset S. Many results concerning these notions are given in this paper. For instance, we present some classes of rings with equivalence between the projectivity and strongly S-projectivity. We provide some cases where a strongly S-hereditary ring is hereditary and others where a hereditary ring is S-Noetherian. Among the main results presented in this paper, there is one concerning the principal ideal domain in which an ideal is strongly S-projective if and only if it's projective and also if and only if it is free. We also show that every short exact sequence $0 \to A \xrightarrow{f} B \to M \to 0$ is split. This means the class of strongly S-projective modules is a subclass of u-S-projective. This result is of great interest because it is used as a tool to prove that the strongly S-projectivity is a local property, as well as allowing other demonstrations.

2. Results

Definition 2.1. *Let R be a ring and S a multiplicative subset of R. An R-module M is called strongly S-projective if there exists a projective submodule P of M such that $sM \subseteq P$ for some $s \in S$.*

From above, we can deduce that every projective module is a strongly S-projective module. But the converse is not true, as the following example will demonstrate.

Example 2.2. *Let $S = \{2^n | n \in \mathbb{N}\}$, a multiplicative set of $\mathbb{Z}$. All $\mathbb{Z}$-modules are projective if only and if they are free. So $\mathbb{Z}/2\mathbb{Z}$ is not projective as $\mathbb{Z}$-module but it is strongly S-projective $\mathbb{Z}$-module.*

Definition 2.3. *Let R be a ring and S a multiplicative subset of R. R is called strongly S-hereditary if every ideal of R is strongly S-projective.*

Recall from [16, Definition 2.1] that an R-module T is said to be a u-S-torsion module if there is an element $s \in S$ such that $sT = 0$.

Remark 2.4. *Let S be a multiplicative subset of a ring R and M an R -module, then:*

(a) *If M is a u-S-torsion, then M is strongly S-projective.*
(b) *If $S \subseteq U(R)$, then M is strongly S-projective if and only if M is projective.*
(c) *If $S \subseteq U(R)$, then R is strongly S-hereditary if and only if R is hereditary.*

Let S be a multiplicative subset of a ring R. The saturation of S is the set $S^* = \{x \in R/xy \in S \text{ for some y} \in R\}$. It is clear that S^* is a multiplicative subset of R and that $S \subseteq S^*$.

Proposition 2.5. *Let R be a ring, S a multiplicative subset of R, and M an R-module. The following statements hold:*

(a) *Let S' be an another multiplicative subset of R. If M is strongly S'-projective, then $S^{-1}M$ is strongly $S^{-1}(S')$-projective as an $S^{-1}R$-module.*
(b) *Let S' be a multiplicative subset of R such that $S \subseteq S'$. If M is strongly S-projective, then M is strongly S'-projective.*
(c) *M is strongly S-projective if and only if M is strongly S^*-projective.*
(d) *If M is strongly S-projective, then $S^{-1}M$ is projective.*

Proof. Let R be a ring, S a multiplicative subset of R and M an R-module.

(a) M is strongly $S^{'}$-projective if and only if there exist an element $s^{'}$ of $S^{'}$ and a projective submodule P of M such that $s^{'}M \subseteq P$. Then $S^{-1}(s^{'}M) \subseteq S^{-1}P$. $\frac{s^{'}}{1}S^{-1}M \subseteq S^{-1}P$. Since P is a projective R-module, $S^{'-1}P$ is a projective $S^{'-1}R$-module. This means that $S^{-1}M$ is strongly $S^{-1}(S^{'})$-projective as an $S^{-1}R$-module.

(b) Let $S^{'}$ be a multiplicative subset of R such that $S \subseteq S^{'}$. Assume that M is strongly S-projective. So, there exists an element s of S and a projective submodule P of M such that $sM \subseteq P$. As $S \subseteq S^{'}$, then $s \in S^{'}$, which means that M is strongly $S^{'}$-projective.

(c) Assume that M is strongly S^{*}-projective. Then there exist an element s of S^{*} and a projective submodule P of M such that $sM \subseteq P$. As $s \in S^{*}$, there exists $t \in R$ such that $ts \in S$. So, $tsM \subseteq tP \subseteq P \subseteq M$. Consequently M is strongly S-projective. The converse results from the second statement.

(d) Let M be an R-module. Then M is strongly S-projective if and only if there exist an element s of S and a projective submodule P of M such that $sM \subseteq P$. Then $S^{-1}(sM) \subseteq S^{-1}P$, so $\frac{s}{1}(S^{-1}M) \subseteq S^{-1}P$. As $\frac{s}{1}$ is a unit in $S^{-1}R$, then $\frac{s}{1}S^{-1}M = S^{-1}M$. Consequently, $S^{-1}M = S^{-1}P$, which means that $S^{-1}M$ is projective.

□

Corollary 2.6. *Let R be a ring and S a multiplicative subset of R. The following statement holds.*

(a) *Let $S^{'}$ be another multiplicative subset of R. If R is strongly $S^{'}$-hereditary, then $S^{-1}R$ is strongly $S^{-1}(S^{'})$-hereditary.*

(b) *R is strongly S-hereditary if and only if R is a strongly S^{*}-hereditary ring.*

(c) *If R is strongly S-hereditary, then $S^{-1}R$ is hereditary.*

Recall from [6], that a ring satisfies DCC_d if for every descending chain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq \cdots$ of ideals of R, there exists $k \in \mathbb{N}$ such that, for each $i \geq k, x_iI_i = I_{i+1}$ for some $x_i \in R$.

Theorem 2.7. *Let R be a strongly S-hereditary ring, where S a multiplicative subset of R. Then:*

(a) *If R a ring which satisfies DCC_d, then for every ideal I of R, there exists an element r of R such that rI is projective.*

(b) *If R is an Artinian ring, then for every ideal I of R, there exists an element s of S such that sI is projective.*

Proof. (a) Let I be an ideal of R. Since R is strongly S-hereditary, I is strongly S-projective. So there exist an element t_1 of S and a projective ideal P_1 such that $t_1I \subseteq P_1 \subseteq I$. Also t_1I is strongly S-projective. Thus there exists a projective ideal P_2 of R and an element t_2 of S such that $t_2t_1I \subseteq P_2 \subseteq t_1I$. Hence, we construct the following chain $t_1t_2I \subseteq P_2 \subseteq t_1I \subseteq P_1 \subseteq I$. Using this process iteratively, we obtain the following chain $\cdots t_nI \subseteq P_n \cdots \subseteq t_2I \subseteq P_2 \subseteq t_1I \subseteq P_1 \subseteq I$. Since R satisfies a DCC_d, there exists an integer i and an element r_i of R such that $r_it_iI = P_i$. As P_i is projective, so is r_it_iI.

(b) Let I be an ideal of R. Since R is strongly S-hereditary, I is strongly S-projective. So there exist an element s_1 of S and a projective ideal P_1 such that $s_1 I \subseteq P_1 \subseteq I$. Also $s_1 I$ is strongly S-projective. Thus there exists a projective ideal P_2 of R and an element s_2 of S such that $s_2 s_1 I \subseteq P_2 \subseteq s_1 I$. Hence, we construct the following chain $s_1 s_2 I \subseteq P_2 \subseteq s_1 I \subseteq P_1 \subseteq I$. Using this process iteratively, we obtain the following chain $\cdots s_1 s_2 \cdots s_n I \subseteq P_n \cdots \subseteq s_1 s_2 I \subseteq P_2 \subseteq s_1 I \subseteq P_1 \subseteq I$. As R is Artinian, this chain is stationary. Thus there exists an integer i such that $t_i I = P_i$ ($t_i = s_1 s_2 \cdots s_n$). Consequently, $t_i I$ is projective.

□

Proposition 2.8. *Let R be a domain and S a multiplicative subset of R. Then the following holds:*

(a) *Every strongly S-projective ideal is S-finite.*
(b) *If R is strongly S-hereditary, then R is S-Noetherian.*

Proof. It suffices to use the definitions of strongly S-projective modules and S-finite ideals. □

Proposition 2.9. *Let R be a ring and S a finite multiplicative subset of R. If R is strongly S-hereditary, then for every ideal I of R, an element s of S exists such that sI is projective. Moreover, if $S \cap Z(I) = \emptyset$, then I is projective.*

Proof. Let I be an ideal of R. By the same method used in the proof of theorem 2.7, we construct the following chain: $\cdots t_n I \subseteq P_n \subseteq \cdots \subseteq t_2 I \subseteq P_2 \subseteq t_1 I \subseteq P_1 \subseteq I$. Since S is finite, two integers i and j exist, such that $t_i = t_j$. Hence, $t_i I = t_j I = P_j$. So $t_i I$ is projective. If $S \cap Z(I) = \emptyset$, then $t_i I$ is isomorphic to I. Moreover, $t_i I$ is projective, so I is projective. □

Corollary 2.10. *Let R be a ring and S a finite multiplicative subset of R such that $S \subseteq Reg(R)$. Then R is strongly S-hereditary if and only if it is hereditary.*

Proof. Since $S \subseteq Reg(R)$, it follows that $S \cap Z(I) = \emptyset$. We also have S a finite multiplicative subset, so if R is strongly S-hereditary, then by using the previous proposition 2.9, every ideal of R is projective, which means that R is hereditary.
The converse is obvious. □

As a direct result, we have the following.

Corollary 2.11. *Let R be a domain and S a finite multiplicative subset of R. Then R is strongly S-hereditary if and only if it is hereditary.*

Proposition 2.12. *Let S be a multiplicative subset of a ring R and M a strongly S-projective R-module. Then the following statements hold:*

(a) *There exist an element s of S, a family $(f_i)_{i \in J}$ of $Hom(M, R)$ and a family of elements $(x_i)_{i \in J}$ of M such that for every $x \in M$: $sx = \Sigma_{i \in J} f_i(x) x_i$ where $f_i(x) = 0$ only for a finite number of $i \in J$.*
(b) *If R is a domain, then every strongly S-projective ideal of R is S-finite.*

Proof. (a) As M is a strongly S-projective module, there exist an element s of S and a projective submodule P of M such that $sM \subseteq P$. Since P is projective, it follows from [15, Theorem 3.15] that there exists a family of homomorphism $(h_i)_{i\in J} \in Hom(P, R)$ satisfying the relationship: $\forall x \in P$, $x = \Sigma_{i\in J} h_i(x)x_i$ where $h_i(x) = 0$ for only a finite number of $i \in J$. For every $i \in J$, we define a homomorphism $f_i \in Hom(M, R)$ by: $f_i(x) = h_i(sx)$ for every x in M. Then, $\forall x \in M$, $sx = \Sigma_{i\in J} h_i(sx)x_i$. So, $\forall x \in M$: $sx = \Sigma_{i\in J} f_i(x)x_i$.

(b) Let I be a strongly S-projective ideal R. Then there exist an element s of S and an projective ideal P such that $sI \subset P \subset I$. As R is a domain and P is projective, P is finitely generated. This means that I is S-finite.

□

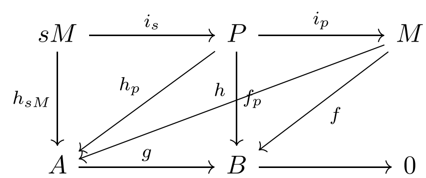
Corollary 2.13. *Let S be a multiplicative subset of a ring R such that $idem(S) = S$ and M be an R-module. Then the following statements are equivalent:*

(a) *M is strongly S-projective.*
(b) *There exists a family of homomorphisms $(f_i)_{i\in J}$ of $Hom(M, R)$ and a family of elements $(x_i)_{i\in J}$ of M such that: $\forall x \in M$, $sx = \Sigma_{i\in J} f_i(x)x_i$ where $f_i(x) = 0$ only for a finite number of $i \in J$*
(c) *sM is projective.*

Proof. $(a) \Rightarrow (b)$: It suffices to apply the above proposition.
$(b) \Rightarrow (c)$: Assume that for every x of M, $sx = \Sigma_{i\in J} f_i(x)x_i$, where $f_i(x) = 0$ only for a finite number of $i \in J$. So, $s^2x = \Sigma_{i\in J} f_i(x)sx_i$, where $f_i(x) = 0$ only for a finite number of $i \in J$. Then $sx = \Sigma_{i\in J} f_i(x)sx_i = \Sigma_{i\in J} f_i(sx)sx_i$, where $f_i(x) = 0$ only for a finite number of $i \in J$. This means that there exist a family $(g_i)_{i\in J}$ of $Hom(sM, R)$ and a family $(y_i)_{i\in J}$ elements of sM such that: $\forall y \in sM$, $y = \Sigma_{i\in J} f_i(y)y_i$, where $g_i(y) = 0$ (g_i is the restriction of f_i to sM and $y_i = sx_i$). So by [15, Theorem 3.15] sM is projective. $(c) \Rightarrow (a)$ is obvious. □

To explore the relationship between the strongly S-projective and the projective modules, the following results were obtained:

Proposition 2.14. *Let S be a multiplicative subset of R and M a strongly S-projective. Then, there exists an element s of S such that for any epimorphism g from an R-module A to another B and for any homomorphism f from M to B, there exists a homomorphism h from M to A such that $sf = goh$*



Proof. Let f be a homomorphism from M to B. Let f_P be the restriction of f to P. Since $f_P \in Hom(P, B)$ and g is an epimorphism from A to B

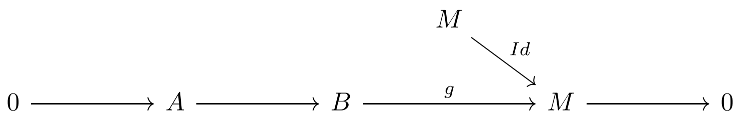
and P is projective, there exists $h_P \in Hom(P, B)$ such that $f_P = goh_P$. Let h_{sM} be the restriction of h_P to sM and h a hommorphism of $Hom(M, A)$ defined as follows: for any $x \in M$, $h(x) = h_{sM}(sx)$. We have $goh(x) = goh_{sM}(sx) = sf(x)$, which means that $sf = goh$. □

Corollary 2.15. *Let S be a multiplicative subset of a ring R and M a strongly S-projective module. For any epimorphism g from an R-module A to another B and for every homomorphism f from M to B satisfying the relationship $f(sx) = f(x)$ for every element s of S, there exists a homomorphism h from M to A such that $f = goh$.*

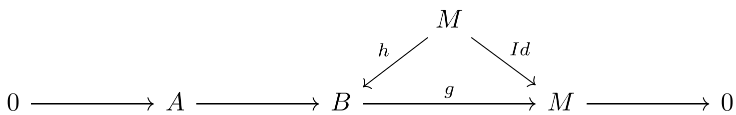
Proof. Applying the same process used in the proof of Proposition 2.14, we obtain a homomorphism h such that $goh(x) = goh_{sM}(sx) = f(sx) = sf(x)$, which means that $goh = sf$. □

Proposition 2.16. *Let S be a multiplicative of a ring R. If M is a strongly S-projective R-module, then every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is u-S-split.*

Proof. Using the above mentioned sequence, we have :



Using Proposition 2.14, we obtain a homomorphism h as shown in the following diagram:



the homomorphism h satisfies the relationship $sId_I = goh$.

□

As a result of the above proposition, we deduce the following important result:

Corollary 2.17. *Let R be a ring. Every strongly S-projective R-module is u-S-projective.*

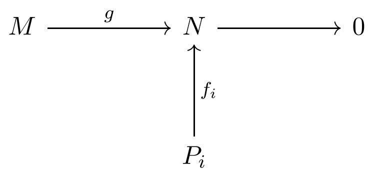
Proof. Let M be strongly S-projective. According to the Proposition 2.16, any short exact sequence $0 \to A \xrightarrow{f} B \to M \to 0$ is split. So, by [17, Theorem 2.5], M is u-S-projective. □

Proposition 2.18. *Let $\{R_i\}_{i\in I}$, $\{M_i\}_{i\in I}$ and $\{S_i\}_{i\in I}$ (S_i be a family of ring, R_i-module and multiplicative subset of R_i) respectively. Let $R = \bigoplus_{i\in I} M_i$, $M = \bigoplus_{i\in I} M_i$ and $S = \bigoplus_{i\in I} S_i$. Then the R-module M is strongly S-projective if and only if for each $i \in I$ M_i is strongly S_i-projective.*

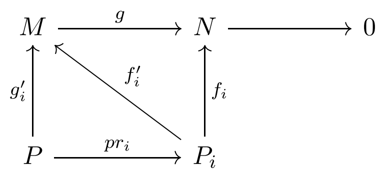
Proof. Assume that for every $i \in I$; M_i is S_i-projective. There exist $s_i \in S_i$ and a projective submodule P_i of M_i such that $s_i M_i \subseteq P_i$. Thus $\bigoplus_{i\in I} s_i M_i \subseteq \bigoplus_{i\in I} P_i \subseteq \bigoplus_{i\in I} M_i$, which means $(s_i)(\oplus_{i\in I} M_i) \subseteq \bigoplus_{i\in I} P_i \subseteq \bigoplus_{i\in I} M_i$ As

$\bigoplus_{i\in I} P_i$ is projective, $\bigoplus_{i\in I} M_i$ is strongly S-projective.

Conversely, if M is strongly S-projective, then a projective submodule P of M exists such that $sM \subseteq P$.So $pr_i(sM) \subseteq pr_i(P) \subseteq pr_i(M)$, where pr_i is the i^{th} projection. So $sM_i \subseteq pr_i(P) \subseteq M_i$. $pr_i(P)$ is a projective submodule of M_i, so M_i is strongly S-projective. It remains to prove that $pr_i(P)$ is a projective. Let the following diagram (where $P_i = pr_i(P)$)



by introducing the module P, we obtain the following diagram:



As P is projective, there exists a homomorphism $g'_i : P \longrightarrow M$ such that $g\circ g'_i = f_i\circ pri$. The homomorphism $g'_i : P_i \longrightarrow M$ is defined by $f'_i(x) = g'_i(y)$, where y is an element of P such that $pr_i(y) = x$. Then $g \circ f'_i = f_i$. Thus P is projective. □

Remark 2.19. *A direct sum of strongly S-projective modules is not necessarily strongly S-projective: Indeed, if $R = \mathbb{Z}$ is the ring of integers, p is a prime in $\mathbb{Z}$ and $S = \{p^n | n \in \mathbb{N}\}$. Let $M_n = \mathbb{Z}/\langle p^n\rangle$, for each $n \geq 1$. Then M_n is strongly projective (because $\forall n \geq 1$, $p^n M_n = 0$). Set $N = \bigoplus_{n=1}^{\infty} M_n$. According to* [17, Example 2.9]*, N is not u-S-projective. So according to Corollary 2.17, N is not strongly S-projective.*

Proposition 2.20. *Let R be a ring and M an R-module. Then the following statements are equivalent:*

(a) *M is projective.*
(b) *M is strongly $\mathfrak{p}$-projective for any $\mathfrak{p} \in spec(R)$.*
(c) *M is strongly $\mathfrak{m}$-projective for any $\mathfrak{m} \in Max(R)$.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ is trivial. $(c) \Rightarrow (a)$: Assume that M is strongly $\mathfrak{m}$-projective for any $\mathfrak{m} \in Max(R)$. Then, M is $\mathfrak{m}$-projective for any $\mathfrak{m} \in Max(R)$. According to [17, Proposition 2.10], M is projective.

□

The following proposition presents a method to construct a strongly S-projective module.

Proposition 2.21. *Let S be a multiplicative subset of a ring R and P a projective ideal of R. Then $(P : s)$ is strongly S-projective.*

Proof. Let S be a multiplicative subset of a ring R and P a projective ideal of R. Then for every $s \in S$ we have: $s(P : s) \subseteq P \subseteq (P : s)$. As P is projective, $(P : s)$ is strongly S-projective. □

Proposition 2.22. *Let R be a principal ideal domain and S a multiplicative subset of R. Then every strongly S-projective torsion-free R-module is free.*

Proof. Let M be a strongly S-projective module over R. Then, there exist an element s of S and a projective submodule P such that $sM \subseteq P \subseteq M$. As R is a principal ideal domain and P is projective, then P is free. Thus sM is free. Since M is torsion-free, sM and M are isomorphic, then M is a free module. □

Proposition 2.23. *Let R and T be two rings such that T is a flat R-module and S a multiplicative subset of R. If M is a strongly S-projective R-module, then $M \otimes_R T$ is a strongly S-projective T-module.*

Proof. Since M is a strongly S-projective R-module, there exists an element s of S and a projective submodule P of M such that $sM \subseteq P$. Since T is a flat R-module, we have $sM \otimes_R T \subseteq P \otimes_R T \subseteq M \otimes_R T$. Thus $M \otimes_R T$ is strongly S-projective since $P \otimes_R T$ is projective. □

Recall from [17] that an R-module M is u-S-semisimple provided that any u-S-short exact sequence $0 \to A \to M \to C \to 0$ is u-S-split.
Recall from [16] that a ring R is u-S-von Neumann regular ring provided that there exists an element $s \in S$ satisfying that for any $a \in R$ there exists $r \in R$ such that $sa = ra^2$.

Proposition 2.24. *Let S be a multiplicative subset of a ring R such that every R-module is strongly S-projective. Then the following are equivalent:*

(a) *R is an u-S-semisimple ring;*
(b) *Every R-module is u-S-projective.*
(c) *R is uniformly S-Noetherian and u-S-von Neumann regular.*

Proof. Assume that every R-module is strongly S-projective. So according to Corollary 2.17 that every R-module is S-projective. Then, using [17, Theorem 3.5], we deduce the previously mentioned statement. □

Remark 2.25. *In the previous proposition, we can easily prove that R cannot have a regular element.*

References

[1] D.D. Anderson, T. Arabaci, U. Tekir, S. Koc, *On S-multiplication modules*, Commun Algebra (2020); 48(8): 3398-3407.
[2] D.D Anderson, S. Chun, *The set of torsion elements of a module*, Commun Algebra (2014); 42(4): 1835-1843.
[3] D.D.Anderson, T.Dumitrescu, S-Noetherian rings, Commun Algebra (2002); 30 (9): 4407-4416.
[4] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading (1969).
[5] Z. Bilgin, L.M. Reyesand, U. Tekir, *On right S-Noetherian rings and S-Noetherian modules*, Commun Algebra (2018); 46(2): 863-869.
[6] R. Dastanpour and A. Ghorbani, *Rings with on chains of ideals*, Commun Algebra (2017), 45(7): 2889-2898.

[7] S. Glaz, *Commutative Coherent Rings*, Springer-Verlag, Lecture Notes in Mathematics, 1371 (1989).
[8] J.A. Huckaba, *Commutative rings with Zero Divisors*, Monographs and Textbooks in Pure and Applied Mathematics 117, Marcel Dekker, Inc., New York, 1988.
[9] A. Hamed and S. Hizem, *Modules satisfying the S-Noetherian property and S-ACCR*, Commun Algebra (2016), 44(7): 1941-1951
[10] A. Hamed and A. Malek, *S-prime ideals of a commutative ring*, Beitr Algebra Geom (2020), 61(3): 533-542.
[11] I. Kaplansky, *Commutative Rings*, Springer Berlin Heidelberg (2006), 153-166.
[12] H. Kim, M. O. Kim, and J. W. Lim, *On S-strong Mori domains*, J. Algebra (2014), 416: 314-332.
[13] J.W. Lim, *A note on S-Noetherian domains*, Kyungpook Math. J. (2015), 55(3): 507-514.
[14] M.D. Larsen and P.J. McCarthy, *Multiplicative Theory of Ideals*, New York: Academic Press. (1971).
[15] J. Rotman, *An Introduction to Homological Algebra*, Second edition, Universitext, Springer, New York, (2009).
[16] X. Zhang, *Characterizing S-flat modules and S-von Neumann regular rings by uniformity*, Bull. Korean Math. Soc. (2022), 59(3): 643-657.
[17] X. Zhanga, W. Qi, *Characterizing S-projective modules and S-semisimple rings by uniformity*, Commun Algebra(2023); 15(1): 139-149.

Laboratory of Modelling and Mathematical Structures, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.
Email address: essafi.oussamaaymane@gmail.com

Laboratory of Modelling and Mathematical Structures, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.
Email address: mahdou@hotmail.com

Department of Mathematics, Marmara University, Istanbul, TURKEY.
Email address: utekir@marmara.edu.tr